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Black hole solutions of Kaluza–Klein supergravity theories and string theory

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Abstract. We find $U(1)_E \times U(1)_M$ non-extremal black hole solutions of six-dimensional Kaluza–Klein supergravity theories. Extremal solutions were found by Cvetič and Youm. Multi-black hole configurations are also presented. After electromagnetic duality transformation is performed, these multi-black hole solutions are mapped into the exact solutions found by Horowitz and Tseytlin in five-dimensional string theory compactified into four dimensions. The massless fields of this theory can be embedded into the heterotic string theory compactified on a 6-torus. Rotating black hole solutions of this string theory can be read off from those of the heterotic string theory found by Sen.

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1. Introduction

There has been considerable interest in the connection between black holes and supersymmetry. The familiar example is the Reissner–Nordström black hole. This can be embedded in $N = 2$ supergravity and the charge of the black hole is identified with the central charge of the extended supersymmetry algebra [4]. Furthermore, using methods similar to those used to prove the positivity of the ADM mass of a gravitational system [5], it was shown that the mass of the black hole is bounded from below by its charge‡. This is exactly the same bound that must be satisfied for black holes to be free of naked singularities. The extremal solutions saturate the mass bound and admit a Killing spinor which is constant with respect to the supercovariant derivative. This condition gives first-order differential equations to be satisfied by the extremal solutions, i.e. Killing spinor equations.

Similar phenomena have been found for other cases as well. One example is the charged black hole arising from string theory [6]. One particular feature of string theory is the non-polynomial coupling of a scalar field to gauge fields. This is also a common characteristic of Kaluza–Klein theories [7]. Those theories have terms like $e^{2\alpha\phi}F_{\mu\nu}F^{\mu\nu}$ in the action where $F_{\mu\nu}$ is the field strength of a gauge field. Charged black hole solutions arising from such theories have attracted much attention because they have drastically different causal structures and thermodynamic properties to the Reissner–Nordström black hole. The $\alpha = 1$ case is the string theory and the supersymmetric properties of the black hole solutions in this case were studied in detail in [8]. Also for $\alpha = \sqrt{3}$ the supersymmetric embedding

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‡ There are some assumptions in proving this result. One of them is that the charge to mass ratio is less than or equal to 1 for any small volume of matter. Even with this assumption it is non-trivial that the black hole mass is bounded below by its charge, since black holes can be formed through a complicated gravitational collapse.

is known and it is five-dimensional Kaluza–Klein supergravity [9]. But it was conjectured that for an arbitrary value of α the corresponding black hole solution admits an embedding in some supergravity theory [10].

One of the motivations of the paper by Cvetič and Youm [1] was to find such embeddings for different values of α using the dimensional reduction of higher-dimensional supergravity theories. They started with the minimal supersymmetric extension of pure gravity in $(4+n)$ dimensions. Keeping only the pure gravity part and performing the dimensional reduction, they obtained the four-dimensional theory with two Abelian gauge fields. It turns out that the resulting bosonic action for each n can be reduced to the action obtained from the six-dimensional (6D) Kaluza–Klein supergravity. They found the extremal black hole configurations using the Killing spinor equations.

One purpose of this paper is to find the black hole solutions of the 6D Kaluza–Klein supergravity by solving the equations of motion directly, thereby obtaining non-extremal black hole solutions as well. This is presented in section 2. Global structures and thermal properties of the black hole solutions are explained. It turns out that the black hole solutions are intimately related with black hole solutions in string theory. We devote sections 3 and 4 to discussing those connections. In section 3 we present multi-black hole solutions of the 6D Kaluza–Klein supergravity theories. After an electromagnetic duality transformation is made, these solutions are mapped into the exact solutions of the 5D string theory compactified into four dimensions. This theory is considered by Horowitz and Tseytlin [2]. We briefly discuss 5D geometry of the exact solutions. In section 4 we show that after a field redefinition, the massless fields of 5D string theory can be embedded into the heterotic string theory. The general electrically charged, rotating black hole solutions are studied by Sen [3]. Thus we can read off the black hole solutions of the 5D string theory from those of the heterotic string theory. In this way, we obtain the rotating black hole solutions of the 5D string theory compactified into four dimensions. Conclusions and speculations are presented in section 5.

2. Black hole solutions of the 6D Kaluza–Klein theory

The bosonic action in $(4+n)$ dimensions is of the form†

$$S_{4+n} = \frac{1}{16\pi G_{4+n}} \int \sqrt{-g^{(4+n)}}\, \mathrm{d}^{4+n}x\, R^{(4+n)}. \tag{1}$$

G_{4+n} is the gravitational constant in $(4+n)$ dimensions. The higher-dimensional metric $g^{(4+n)}_{AB}$ is taken as

$$g^{(4+n)}_{AB} = \begin{pmatrix} \mathrm{e}^{-\psi/\alpha} g_{\lambda\pi} + \mathrm{e}^{2\psi/n\alpha}\rho_{\tilde\lambda\tilde\pi} A^{\tilde\lambda}_{\lambda} A^{\tilde\pi}_{\pi} & \mathrm{e}^{2\psi/n\alpha}\rho_{\tilde\lambda\tilde\pi} A^{\tilde\lambda}_{\lambda} \\ \mathrm{e}^{2\psi/n\alpha}\rho_{\tilde\lambda\tilde\pi} A^{\tilde\pi}_{\pi} & \mathrm{e}^{2\psi/n\alpha}\rho_{\tilde\lambda\tilde\pi} \end{pmatrix}. \tag{2}$$

Greek lowercase letters denote the spacetime indices in four dimensions while lowercase letters with a tilde are used for the internal coordinates. $\rho_{\tilde\lambda\tilde\pi}$ satisfies $\det\rho_{\tilde\lambda\tilde\pi} = 1$, i.e. $\rho_{\tilde\lambda\tilde\pi}$ is the unimodular part of the internal metric. All fields have no dependence on the internal coordinates. It is shown in [1] that if all the gauge fields are Abelian, the supersymmetric configuration is possible only if the internal group $U(1)^n$ is broken down to $U(1)_E \times U(1)_M$.

† With regard to the metric sign and the definition of the curvature, we follow the convention used by Misner *et al* [12]. The metric signature is $(-+\cdots+)$, $R^{\alpha}_{\ \beta\gamma\delta} = \partial_\gamma \Gamma^{\alpha}_{\beta\gamma\delta} + \cdots$, $R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu}$.

Thus electric charge and magnetic charge should be associated to different $U(1)$ sectors. The internal metric $\rho_{\tilde{\alpha}\tilde{\beta}}$ is assumed to be

$$\rho_{\tilde{\alpha}\tilde{\beta}} = \mathrm{diag}\left(\rho_1, \ldots, \rho_{n-1}, \prod_{k=1}^{n-1} \rho_k\right). \tag{3}$$

Then the resulting action in four dimensions after the trivial integration over the internal coordinates is

$$S_4 = \frac{1}{16\pi G} \int \sqrt{-g}\, \mathrm{d}^4 x \Bigg(R - \tfrac{1}{4} \mathrm{e}^{\alpha\psi} \rho_{n-1} \big(F_{\mu\nu}^{n-1}\big)^2 - \tfrac{1}{4} \mathrm{e}^{\alpha\psi} \rho_n \big(F_{\mu\nu}^{n}\big)^2 \\ -\tfrac{1}{2} \partial_\mu \psi \partial^\mu \psi - \tfrac{1}{4} \sum_{i=1}^{n} \partial_\mu \log \rho_i \partial^\mu \log \rho_i \Bigg), \tag{4}$$

where $\alpha = \sqrt{(n+2)/n}$. The gravitational coupling constant G in four dimensions is given by $G = G_{n+4}(2\pi R)^n$ for a toroidal compactification where each internal dimension has radius R, while $F_{\mu\nu}^{n-1}$ and $F_{\mu\nu}^{n}$ denote the unbroken Abelian gauge group. This action can be reduced to that of 6D Kaluza–Klein theory with the field redefinition

$$\begin{aligned} &\phi \equiv \frac{1}{\sqrt{2}\alpha} \psi, \qquad \chi_i \equiv \frac{1}{2\sqrt{2}} \left(\log \rho_i + \frac{2}{n\alpha} \psi \right), \qquad i = 1, \ldots, n-2 \\ &\chi_{n-1} \equiv \frac{1}{2\sqrt{2}} \left(\log \rho_{n-1} + \frac{2-n}{n\alpha} \psi \right), \qquad \chi_n \equiv \frac{1}{2\sqrt{2}} \left(\log \rho_n + \frac{2-n}{n\alpha} \psi \right). \end{aligned} \tag{5}$$

Then the corresponding action is

$$S_4 = \frac{1}{16\pi G} \int \sqrt{-g}\, \mathrm{d}^4 x \Bigg(R - \tfrac{1}{4} \mathrm{e}^{2\sqrt{2}(\phi + \chi_{n-1})} (F_{n-1})^2 - \tfrac{1}{4} \mathrm{e}^{2\sqrt{2}(\phi + \chi_n)} (F_n)^2 \\ -2 \sum_{i=1}^{n} (\nabla \chi_i)^2 - 2\partial_\mu \phi \big(\partial^\mu \chi_{n-1} + \partial^\mu \chi_n \big) - 2(\nabla\phi)^2 \Bigg). \tag{6}$$

We see that the field equations derived from this action will admit a solution with χ_i set to a constant for $i = 1, \ldots, n-2$. This implies that $\chi_{n-1} + \chi_n$ is also constant since $\sum_{i=1}^{n} \chi_i = 0$. Absorbing such constants into field redefinition of the gauge fields and defining $\chi \equiv \chi_{n-1}$, $K_{\mu\nu} \equiv F_{\mu\nu}^{n-1}$, $F_{\mu\nu} \equiv F_{\mu\nu}^{n}$, we finally obtain the following action†:

$$S = \frac{1}{16\pi} \int \sqrt{-g}\, \mathrm{d}^4 x \big(R - 2(\nabla\phi)^2 - 4(\nabla\chi)^2 - \mathrm{e}^{2\sqrt{2}(\phi+\chi)} K^2 - \mathrm{e}^{2\sqrt{2}(\phi-\chi)} F^2 \big). \tag{7}$$

If the metric in four dimensions takes the form

$$\mathrm{d}s^2 = -A^2(r)\, \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{A^2(r)} + R^2(r)\, \mathrm{d}\Omega^2, \tag{8}$$

the equations for the gauge fields are solved by

$$K_{\theta\phi} = Q_m \sin\theta, \qquad F^{rt} = \frac{Q_e}{R^2 \mathrm{e}^{2\sqrt{2}(\phi-\chi)}}. \tag{9}$$

Then the equations of the two scalar fields are

$$\frac{1}{R^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(R^2 A^2 \frac{\mathrm{d}\phi}{\mathrm{d}r} \right) = \sqrt{2}\, \mathrm{e}^{2\sqrt{2}(\phi+\chi)} \frac{Q_m^2}{R^4} - \sqrt{2}\, \mathrm{e}^{-2\sqrt{2}(\phi-\chi)} \frac{Q_e^2}{R^4}, \tag{10}$$

$$\frac{1}{R^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(R^2 A^2 \frac{\mathrm{d}\chi}{\mathrm{d}r} \right) = \frac{\sqrt{2}}{2} \mathrm{e}^{2\sqrt{2}(\phi+\chi)} \frac{Q_m^2}{R^4} + \frac{\sqrt{2}}{2} \mathrm{e}^{-2\sqrt{2}(\phi-\chi)} \frac{Q_e^2}{R^4}. \tag{11}$$

† From now on we set $G = c = 1$.

From the gravitational field equations, we can obtain the following combinations:

$$R^2\big(G_r^r + G_\theta^\theta\big) = \tfrac{1}{2}\big(A^2R^2\big)'' - 1 = 0, \tag{12}$$

$$R^2\big(2G_\theta^\theta + G_r^r - G_t^t\big) = \big(\big(A^2\big)'R^2\big)' = 2\mathrm{e}^{2\sqrt{2}(\phi+\chi)}\frac{Q_m^2}{R^4} + 2\mathrm{e}^{-2\sqrt{2}(\phi-\chi)}\frac{Q_e^2}{R^4}, \tag{13}$$

where $'$ denotes differentiation by r. Since we are searching for solutions with a regular horizon, we can set $A^2R^2 = (r - r_+)(r - r_-)$ with $r_+ \geqslant r_-$ where $r = r_+$ is the location of the horizon. If we define

$$\rho \equiv \frac{1}{r_+ - r_-}\log\left(\frac{r - r_+}{r - r_-}\right), \tag{14}$$

then we can write

$$A^2R^2\frac{\mathrm{d}}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}\rho}. \tag{15}$$

Throughout this section ρ is used as a short notation for the more complicated expression of r. Comparing equations (11) and (13) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}2\sqrt{2}\chi = \frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\log A^2, \tag{16}$$

$$2\sqrt{2}\chi = \log A^2 + \beta\rho + 2\sqrt{2}\chi_\infty. \tag{17}$$

Here β is a constant and χ_∞ is the asymptotic value of χ. We require that χ be regular and finite at the horizon. Thus $\beta = -(r_+ - r_-)$ and

$$\mathrm{e}^{2\sqrt{2}\chi} = A^2\frac{r - r-}{r - r+}\mathrm{e}^{2\sqrt{2}\chi_\infty}, \tag{18}$$

$$R^2\mathrm{e}^{2\sqrt{2}\chi} = (r - r_-)^2\mathrm{e}^{2\sqrt{2}\chi_\infty}, \tag{19}$$

Finally, the equations for the scalar fields can be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left(\frac{2\sqrt{2}\phi + 4\sqrt{2}\chi}{2}\right) = 4Q_m^2\mathrm{e}^{-2\sqrt{2}\chi_\infty}\frac{r - r_+}{r - r_-}\mathrm{e}^{2\sqrt{2}\phi + 4\sqrt{2}\chi}, \tag{20}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left(\frac{-2\sqrt{2}\phi + 4\sqrt{2}\chi}{2}\right) = 4Q_e^2\mathrm{e}^{-2\sqrt{2}\chi_\infty}\frac{r - r_+}{r - r_-}\mathrm{e}^{-2\sqrt{2}\phi + 4\sqrt{2}\chi}. \tag{21}$$

Equations (20) and (21) can be solved by

$$\begin{aligned} \exp\frac{2\sqrt{2}(\phi - \phi_\infty) + 4\sqrt{2}(\chi - \chi_\infty)}{2} &= \frac{r - r_-}{r + B}, \\ \exp\frac{-2\sqrt{2}(\phi - \phi_\infty) + 4\sqrt{2}(\chi - \chi_\infty)}{2} &= \frac{r - r_-}{r + C}. \end{aligned} \tag{22}$$

It is easily checked that this ansatz works. The ansatz is motivated by studying $Q_e = 0$ case where the equations to be solved are identical to those of black holes of the five-dimensional Kaluza–Klein supergravity after a suitable field redefinition. In solving the latter, we encounter the same equation as (20) and (21). Inserting (22) into (18) and (19), we obtain the expression for A^2 and R^2 in terms of r.

We can choose $C = -B = r_0$ by a suitable shift of $r \to r + \Gamma$. Thus the black hole solutions are given by

$$A^2 = \frac{r - r_+}{\sqrt{r^2 - r_0^2}}, \qquad R^2 = (r - r_-)\sqrt{r^2 - r_0^2},$$
$$\mathrm{e}^{2\sqrt{2}\phi} = \mathrm{e}^{2\sqrt{2}\phi_\infty}\frac{r + r_0}{r - r_0}, \qquad \mathrm{e}^{2\sqrt{2}\chi} = \mathrm{e}^{2\sqrt{2}\chi_\infty}\frac{r - r_-}{\sqrt{r^2 - r_0^2}}, \tag{23}$$
$$F^{rt} = \frac{Q_e}{(r + r_0)^2}\mathrm{e}^{-2\sqrt{2}(\phi_\infty - \chi_\infty)}, \qquad K_{\theta\phi} = Q_m \sin\theta.$$

The constraints on r_+, r_-, r_0 are

$$(r_+ + r_0)(r_- + r_0) = 4Q^2, \qquad (r_+ - r_0)(r_- - r_0) = 4P^2. \tag{24}$$

Here P and Q are defined as

$$Q^2 \equiv Q_e^2 \mathrm{e}^{-2\sqrt{2}\phi_\infty + 2\sqrt{2}\chi_\infty}, \qquad P^2 \equiv Q_m^2 \mathrm{e}^{2\sqrt{2}\phi_\infty + 2\sqrt{2}\chi_\infty}. \tag{25}$$

From the solution (23), we see that $r_+ = 2M$ where M is the mass of a black hole, $r_0 = \sqrt{2}\Sigma$ where Σ is the charge of the scalar ϕ, and $r_- = -2\Delta$ where Δ is the charge of χ. The charge Σ is defined by

$$\phi = \phi_\infty + \frac{\Sigma}{r} + \mathrm{O}\left(\frac{1}{r^2}\right), \qquad r \to \infty. \tag{26}$$

On the other hand, Δ is defined by

$$\sqrt{2}\chi = \sqrt{2}\chi_\infty + \frac{\Delta}{r} + \mathrm{O}\left(\frac{1}{r^2}\right), \qquad r \to \infty, \tag{27}$$

since we adopt the different normalization for ϕ and χ in action (7). Clearly, Σ and Δ are not independent parameters and depend on M, P and Q. Their dependence is given by cubic equations but those equations are not particularly illuminating. There are seven parameters appearing in the solution (23) and there are two constraints given by (24) among those parameters. Hence the black hole solutions are parametrized by five parameters, which we can take as mass, electric charge, magnetic charge and asymptotic values of scalar fields. From equation (24), one can easily find that $M \geqslant (|Q|+|P|)/2$ and $|\Sigma| \leqslant (||Q|-|P||)/\sqrt{2}$. The sign of Σ is the same as that of $|Q| - |P|$. Also $r_+ \geqslant r_- \geqslant |r_0|$. Calculation of the curvature tensors indicates that $r = r_+$ is indeed a regular horizon and $r = r_-$ is a curvature singularity. Comparing with the Reissner–Nordström black hole, the would-be inner horizon turns into the singularity. The extremal solutions where r_+ coincides with r_- agree with the solutions found by Cvetič and Youm [1]. Note that $M = (|Q|+|P|)/2$, $\Sigma = (|Q|-|P|)/\sqrt{2}$ and $\Delta = -M$ so that $M^2 + \Sigma^2 + \Delta^2 = Q^2 + P^2$ in the extremal case. This is the force balance condition, as we will see later.

The causal structure of the non-extremal case is that of the Schwarzschild black hole. For extremal black holes, the situation is twofold. If both electric and magnetic charge are non-zero, the corresponding black hole has a null singularity. This can be seen from the fact that the radial null geodesics satisfy $\pm\mathrm{d}t \propto \mathrm{d}r/(r - r_+)$, which implies that as $r \to r_+$, the geodesics reach arbitrarily large values of $|t|$. This shows that an outgoing null geodesic must cross every ingoing null geodesic. However, if either of the charges is zero, the singularity becomes timelike naked. In this case $r_+ = r_- = \pm r_0$ and $\pm\mathrm{d}t \propto \mathrm{d}r/\sqrt{r - r_+}$ for radial null rays near $r = r_+$.

The temperature of the black hole can be found from the periodicity of its Euclidean continuation [11], or alternatively from its surface gravity. It is given by $T =$

$1/4\pi\sqrt{r_+^2 - r_0^2}$. In the extremal case, $r_+ = |Q| + |P|$ and $r_0 = |Q| - |P|$. Hence T approaches $1/8\pi\sqrt{|P||Q|}$ in the extremal limit. The entropy of the black holes can be evaluated in two ways. One may integrate the first law of thermodynamics. Or one can calculate the thermodynamic functions directly, using the saddle-point approximation for the action of the black holes in the Euclidean continuation [11]. Either way gives $S = \frac{1}{4}A = \pi(r_+ - r_-)\sqrt{r_+^2 - r_0^2}$. In the extremal limit, the black holes approach zero entropy and non-zero temperature configurations.

So far we have presented the black hole solutions of 6D Kaluza–Klein supergravity. For other dimensions we can read off the result from equation (5). However, the same metric remains a solution for all dimensions.

3. Multi-black hole solutions and their string interpretation

Now we will look for a static solution representing a collection of extremal black holes, with the following ansatz for the metric in isotropic coordinates:

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}\,\mathrm{d}t^2 + \mathrm{e}^{-2U}(\mathrm{d}x_i)^2. \tag{28}$$

The non-zero components of the Ricci tensor in the coordinate basis are

$$R_{00} = \mathrm{e}^{4U}\partial_i\partial_i U \qquad R_{ij} = -2\partial_i U\partial_j U + \delta_{ij}\partial_k\partial_k U. \tag{29}$$

If we choose the gauge fields as

$$F_{i0} = \partial_i\Psi, \qquad K_{ij} = \varepsilon_{ijk}\mathrm{e}^{-2\sqrt{2}(\phi+\chi)-2U}\partial_k\lambda, \tag{30}$$

the equations of motion and Bianchi identities give

$$\partial_i\left(\mathrm{e}^{2\sqrt{2}(\phi-\chi)-2U}\partial_i\Psi\right) = 0, \qquad \partial_i\left(\mathrm{e}^{-2\sqrt{2}(\phi+\chi)-2U}\partial_i\lambda\right) = 0 \tag{31}$$

and the scalar field equations are

$$\begin{aligned}\partial_i\partial_i\phi &= \sqrt{2}\,\mathrm{e}^{-2\sqrt{2}(\phi+\chi)-2U}(\partial_i\lambda)^2 - \sqrt{2}\,\mathrm{e}^{2\sqrt{2}(\phi-\chi)-2U}(\partial_i\Psi)^2,\\ \partial_i\partial_i\chi &= \frac{\sqrt{2}}{2}\mathrm{e}^{-2\sqrt{2}(\phi+\chi)-2U}(\partial_i\lambda)^2 + \frac{\sqrt{2}}{2}\mathrm{e}^{2\sqrt{2}(\phi-\chi)-2U}(\partial_i\Psi)^2.\end{aligned} \tag{32}$$

The gravitational field equations give

$$\partial_i\partial_i U = \mathrm{e}^{-2\sqrt{2}(\phi+\chi)-2U}(\partial_i\lambda)^2 + \mathrm{e}^{2\sqrt{2}(\phi-\chi)-2U}(\partial_i\Psi)^2, \tag{33}$$

$$\partial_i U\partial_j U = -\partial_i\phi\partial_j\phi - 2\partial_i\chi\partial_j\chi + \mathrm{e}^{-2\sqrt{2}(\phi+\chi)-2U}\partial_i\lambda\partial_j\lambda + \mathrm{e}^{2\sqrt{2}(\phi-\chi)-2U}\partial_i\Psi\partial_j\Psi. \tag{34}$$

The relevant solutions of these equations are

$$\begin{aligned}&\sqrt{2}\chi = U + \sqrt{2}\chi_\infty,\\ &\mathrm{e}^{-\sqrt{2}(\phi+2\chi)} = H_1, \qquad \mathrm{e}^{-\sqrt{2}(-\phi+2\chi)} = H_2,\\ &2\mathrm{e}^{\sqrt{2}\chi_\infty}\lambda = \pm\frac{1}{H_1}, \qquad 2\mathrm{e}^{\sqrt{2}\chi_\infty}\Psi = \pm\frac{1}{H_2},\\ &\partial_i\partial_i H_1 = 0, \qquad \partial_i\partial_i H_2 = 0.\end{aligned} \tag{35}$$

One particular solution representing the multiblack hole configuration is given by

$$\begin{aligned}H_1 &= \mathrm{e}^{-\sqrt{2}(\phi_\infty+2\chi_\infty)}\left(1 + \sum_{i=1}^{n}\frac{2|P_i|}{|x - x_i|}\right),\\ H_2 &= \mathrm{e}^{\sqrt{2}(\phi_\infty-2\chi_\infty)}\left(1 + \sum_{i=1}^{n}\frac{2|Q_i|}{|x - x_i|}\right).\end{aligned} \tag{36}$$

One can easily see that this is the collection of n extremal black holes. The relation between the parameters of each black hole is

$$M_i = -\Delta_i = \frac{|Q_i| + |P_i|}{2}, \qquad \Sigma_i = \frac{|Q_i| - |P_i|}{\sqrt{2}}. \tag{37}$$

This condition implies the force balance. To see this explicitly, let us consider gravitational, electromagnetic, and scalar forces. The force between two distant objects of masses and charges $(M_1, Q_1, P_1, \Sigma_1, \Delta_1)$ and $(M_2, Q_2, P_2, \Sigma_2, \Delta_2)$ is given by

$$F_{12} = -\frac{M_1 M_2}{r_{12}^2} + \frac{Q_1 Q_2}{r_{12}^2} + \frac{P_1 P_2}{r_{12}^2} - \frac{\Sigma_1 \Sigma_2}{r_{12}^2} - \frac{\Delta_1 \Delta_2}{r_{12}^2}. \tag{38}$$

The scalar forces are attractive for charges of the same type and repulsive for charges of opposite sign. Using equation (37), we see that F_{12} vanishes. This force balance allows the black holes to be located at any place, in equilibrium with the other black holes.

Interestingly enough, the above solutions are related to the exact string solutions found by Horowitz and Tseytlin [2] by the electromagnetic dual transformation. From the action (7), if we perform the duality transformation

$$K_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} C^{\lambda\sigma} \mathrm{e}^{-2\sqrt{2}(\phi+\chi)}, \tag{39}$$

the resulting action is

$$S_1 = \frac{1}{16\pi}\int \sqrt{-g}\, \mathrm{d}^4x \big(R - 2(\nabla\phi)^2 - 4(\nabla\chi)^2 - \mathrm{e}^{-2\sqrt{2}(\phi+\chi)} C^2 - \mathrm{e}^{2\sqrt{2}(\phi-\chi)} F^2\big), \tag{40}$$

where $\varepsilon_{\mu\nu\lambda\sigma}$ is an antisymmetric tensor with $\varepsilon_{1234} = 1$. Now if we define

$$\begin{aligned} &F^s_{\mu\nu} \equiv 2F_{\mu\nu}, \qquad B^s_{\mu\nu} \equiv 2C_{\mu\nu}, \\ &\varphi \equiv 2\sqrt{2}\chi, \qquad \sigma \equiv \sqrt{2}\phi, \end{aligned} \tag{41}$$

the action is written as

$$S_2 = \frac{1}{16\pi}\int \sqrt{-g}\, \mathrm{d}^4x \big(R - (\nabla\sigma)^2 - \tfrac{1}{2}(\nabla\varphi)^2 - \tfrac{1}{4}\mathrm{e}^{-2\sigma-\varphi}\big(B^s_{\mu\nu}\big)^2 - \tfrac{1}{4}\mathrm{e}^{2\sigma-\varphi}\big(F^s_{\mu\nu}\big)^2\big). \tag{42}$$

This is the dimensional reduction of the 5D bosonic string action. To see this, we start with the leading-order term in 5D bosonic string action

$$S_5 = \kappa^0 \int \mathrm{d}^5x\, \sqrt{-g_{(5)}}\, \mathrm{e}^{-2\phi_s}\big(R + 4(\nabla\phi_s)^2 - \tfrac{1}{12}(H_{MNK})^2 + \mathrm{O}(\alpha')\big), \tag{43}$$

where κ^0 is $H_{MNK} = 3\partial_{[M} B'_{NK]} = \partial_M B'_{NK} + \partial_K B'_{MN} + \partial_N B'_{KM}$. Here $\partial_{[M} B'_{NK]}$ is an antisymmetric third rank tensor of strength 1†. Assuming that all the fields are independent of x^5, we obtain the 4D reduced action

$$\begin{aligned} \tilde{S}_4 = \hat{\kappa}_0 \int \mathrm{d}^4x\, \sqrt{-g}\, \mathrm{e}^{-2\phi_s+\sigma}\big(&\hat{R} + 4(\partial_\mu\phi_s)^2 - 4\partial_\mu\phi_s\partial^\mu\sigma \\ &- \tfrac{1}{12}(\hat{H}_{\mu\nu\lambda})^2 - \tfrac{1}{4}\mathrm{e}^{2\sigma}\big(F^s_{\mu\nu}\big)^2 - \tfrac{1}{4}\mathrm{e}^{-2\sigma}\big(B^s_{\mu\nu}\big)^2 + \mathrm{O}(\alpha')\big) \end{aligned} \tag{44}$$

where

$$\begin{aligned} &g_{55} \equiv \mathrm{e}^{2\sigma}, \qquad F^s_{\mu\nu} = \partial_\mu A^s_\nu - \partial_\nu A^s_\mu, \\ &B^s_{\mu\nu} = \partial_\mu B^s_\nu - \partial_\nu B^s_\mu, \qquad A^s_\mu \equiv g_{\mu 5}\mathrm{e}^{-2\sigma}, \\ &B^s_\mu \equiv B'_{\mu 5}, \qquad \hat{H}_{\lambda\mu\nu} = 3\partial_{[\lambda} B'_{\mu\nu]} - 3A^s_{[\lambda} B^s_{\mu\nu]}. \end{aligned} \tag{45}$$

† From now on we denote an antisymmetric nth rank tensor of strength 1 as $A_{[\mu_1\mu_2\cdots\mu_n]}$.

The 5D metric is given in terms of the 4D metric as

$$\mathrm{d}s^2 = \mathrm{e}^{2\sigma}\big(\mathrm{d}x^5 + A_t\,\mathrm{d}t\big)^2 + g_{\alpha\beta}\,\mathrm{d}x^{\alpha\beta}. \tag{46}$$

Setting $\varphi = 2\phi - \sigma$ and using the Einstein metric $g^E_{\alpha\beta} = \mathrm{e}^{-\varphi} g_{\alpha\beta}$, we obtain

$$S'_4 = \hat{\kappa}_0 \int \sqrt{-g_E}\,\mathrm{d}^4x\big(R_E - (\nabla\sigma)^2 - \tfrac{1}{2}(\nabla\varphi)^2 - \tfrac{1}{12}\mathrm{e}^{-2\varphi}(\hat{H}_{\mu\nu\lambda})^2 \\ -\tfrac{1}{4}\mathrm{e}^{-2\sigma-\varphi}\big(B^s_{\mu\nu}\big)^2 - \tfrac{1}{4}\mathrm{e}^{2\sigma-\varphi}\big(F^s_{\mu\nu}\big)^2 + \mathrm{O}(\alpha')\big). \tag{47}$$

This is equal to S_2 of (42) if we set $\hat{H}_{\lambda\mu\nu} = 0$.

The multi-black hole solutions found at (35) and (36) are transformed into the solutions found by Horowitz and Tseytlin (equation (4.6) in their paper [2]). Those solutions are exact solutions to all orders in α'. These bosonic solutions are also shown to be exact in the closed superstring and in the heterotic string theory as well.

4. Rotating black hole solutions of the five-dimensional string theory compactified into four dimensions

There is a close relationship between the five-dimensional string theory compactified into four dimensions, and four-dimensional heterotic string theory with toroidal compactification. If we define $\Phi = 2\phi_s - \sigma$, then $\tilde{S}_4$ in (44) can be written as

$$\tilde{S}_4 = \int \mathrm{d}^4x\,\sqrt{-g}\,\mathrm{e}^{-\Phi}\big(R + (\partial_\mu\Phi)^2 - (\partial_\mu\sigma)^2 - \tfrac{1}{4}\mathrm{e}^{2\sigma}\big(F^s_{\mu\nu}\big)^2 - \tfrac{1}{4}\mathrm{e}^{-2\sigma}\big(B^s_{\mu\nu}\big)^2\big) \tag{48}$$

On the other hand, the massless fields in heterotic string theory compactified on a six-dimensional torus consists of the metric $g_{\mu\nu}$, the antisymmetric tensor field $B_{\mu\nu}$, 28 $U(1)$ gauge fields $A^{(a)}_\mu$ $(1 \leqslant a \leqslant 28)$, the scalar dilaton field Φ, and a 28×28 symmetric matrix valued scalar field M satisfying

$$MLM^T = L, \qquad M^T = M. \tag{49}$$

Here L is a 28×28 symmetric matrix with 22 eigenvalues -1 and six eigenvalues $+1$. For definiteness we will take L to be

$$L = \begin{pmatrix} -I_{22} & \\ & I_6 \end{pmatrix}, \tag{50}$$

where I_n denotes an $n \times n$ identity matrix. The action describing the effective field theory of these massless bosonic fields is given by [3],

$$S = C\int \mathrm{d}^4x\,\sqrt{-g}\,\mathrm{e}^{-\Phi}\big(R + (\partial_\mu\phi)^2 + \tfrac{1}{8}\,\mathrm{Tr}\,\big(\partial_\mu ML\partial^\mu L\big) - F^{(a)}_{\mu\nu}(LML)_{ab}F^{\mu\nu(b)} \\ -\tfrac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\big), \tag{51}$$

where

$$\begin{aligned} F^{(a)}_{\mu\nu} &= \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu, \\ H_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} + 6A^{(a)}_{[\mu}F^{(b)}_{\nu\rho]}L_{ab}. \end{aligned} \tag{52}$$

If we choose the special M

$$M = \begin{pmatrix} \cosh 2\sigma & & \sinh 2\sigma & \\ & -I_{21} & & \\ \sinh 2\sigma & & \cosh 2\sigma & \\ & & & I_5 \end{pmatrix}, \tag{53}$$

then S can be reduced to

$$S_4 = \int \mathrm{d}^4x \sqrt{-g}\, \mathrm{e}^{-\Phi} \Bigg(R + (\partial_\mu \Phi)^2 - (\partial_\mu \sigma)^2 - \mathrm{e}^{2\sigma} \left(\frac{F^{(1)}_{\mu\nu} - F^{(23)}_{\mu\nu}}{\sqrt{2}} \right)^2 - \mathrm{e}^{-2\sigma} \left(\frac{F^{(1)}_{\mu\nu} + F^{(23)}_{\mu\nu}}{\sqrt{2}} \right)^2 \Bigg). \tag{54}$$

Thus if we set

$$F^s_{\mu\nu} = \sqrt{2}\big(F^{(1)}_{\mu\nu} - F^{(23)}_{\mu\nu}\big), \qquad B^s_{\mu\nu} = \sqrt{2}\big(F^{(1)}_{\mu\nu} + F^{(23)}_{\mu\nu}\big), \tag{55}$$

we recover the action (48). Hence we can read off the black hole solutions of (48) from Sen's results, which construct the general electrically charged rotating black hole solutions of the heterotic string theory.

But there is a little difference in the definition of the antisymmetric tensor fields between Sen's work and Horowitz and Tseytlin's. Following Sen's definition (52), we obtain

$$H_{\mu\nu\lambda} = 3\partial_{[\mu} B_{\nu\lambda]} - \tfrac{3}{2} A^s_{[\mu} B^s_{\nu\lambda]} - \tfrac{3}{2} B^s_{[\mu} F^s_{\nu\lambda]}, \tag{56}$$

while Horowitz and Tseytlin use

$$\hat{H}_{\mu\nu\lambda} = 3\partial_{[\mu} B'_{\nu\lambda]} - 3A^s_{[\mu} B^s_{\nu\lambda]}. \tag{57}$$

If we set $B'_{\mu\nu} = B_{\mu\nu} - A^s_{[\mu} B^s_{\nu]}$, $H_{\mu\nu\lambda}$ is equal to $\hat{H}_{\mu\nu\lambda}$. Thus the two definitions differ by field redefinition and this redefinition does not change the gauge-invariant field strength. One can check that equations of motion do not change under the above field redefinition. The rotating black hole solutions are given by

$$\begin{aligned}
\mathrm{d}s_E^2 = &-\frac{\rho^2 + a^2\cos^2\theta - 2m\rho}{\sqrt{\Delta}}\mathrm{d}t^2 + \frac{\sqrt{\Delta}}{\rho^2 + a^2 - 2m\rho}\mathrm{d}\rho^2 + \sqrt{\Delta}\,\mathrm{d}\theta^2 \\
&+ \frac{\sin^2\theta}{\sqrt{\Delta}}\big[\Delta + a^2\sin^2\theta\big(\rho^2 + a^2\cos^2\theta + 2m\rho\cosh\alpha\cosh\beta\big)\big]\mathrm{d}\phi^2 \\
&- \frac{2}{\sqrt{\Delta}} m\rho a\sin^2\theta(\cosh\alpha + \cosh\beta)\,\mathrm{d}t\,\mathrm{d}\phi,
\end{aligned}$$

where $\mathrm{d}s_E^2 = \mathrm{e}^{-\Phi} g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$ is the Einstein metric and

$$\Delta = \big[\rho^2 + a^2\cos^2\theta + m\rho(\cosh\alpha\cosh\beta - 1)\big]^2 - m^2\rho^2\sinh^2\alpha\sinh^2\beta. \tag{58}$$

The other fields are given by

$$\begin{aligned}
\exp\Phi &= \frac{\rho^2 + a^2\cos^2\theta}{\Delta} \\
\exp\sigma &= \frac{\rho^2 + a^2\cos^2\theta + m\rho(\cosh\alpha\cosh\beta - \sinh\alpha\sinh\beta - 1)}{\Delta}
\end{aligned} \tag{59}$$

$$\begin{aligned}
A^s_t = &-\Delta^{-1} m\rho\big[\big(\rho^2 + a^2\cos^2\theta\big)(\cosh\beta\sinh\alpha - \cosh\alpha\sinh\beta) \\
&+ m\rho(\cosh\alpha - \cosh\beta)(\sinh\alpha + \sinh\beta)\big] \\
A^s_\phi = &\;\Delta^{-1} m\rho a\sin^2\theta\big[\big(\rho^2 + a^2\cos^2\theta\big)(\sinh\alpha - \sinh\beta) \\
&+ m\rho(\cosh\alpha - \cosh\beta)(\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha)\big],
\end{aligned} \tag{60}$$

$$
\begin{aligned}
B_t^s &= -\Delta^{-1} m\rho \big[\big(\rho^2 + a^2 \cos^2\theta\big)(\cosh\beta \sinh\alpha + \cosh\alpha \sinh\beta) \\
&\qquad + m\rho(\cosh\alpha - \cosh\beta)(\sinh\alpha - \sinh\beta)\big], \\
B_\phi^s &= \Delta^{-1} m\rho a \sin^2\theta \big[\big(\rho^2 + a^2 \cos^2\theta\big)(\sinh\alpha + \sinh\beta) \\
&\qquad + m\rho(\cosh\alpha - \cosh\beta)(\sinh\alpha \cosh\beta - \sinh\beta \cosh\alpha)\big], \\
B_{t\phi} &= \Delta^{-1} m\rho a \sin^2\theta (\cosh\alpha - \cosh\beta)\big[\big(\rho^2 + a^2\cos^2\theta\big) + m\rho(\cosh\alpha\cosh\beta - 1)\big],
\end{aligned}
\tag{61}
$$

and $B'_{\mu\nu} = B_{\mu\nu} - A^s_{[\mu} B^s_{\nu]}$. The various properties of the above solutions are studied in [3]. Non-extremal solutions with non-zero angular momentum have two horizons and their global structure are similar to that of the Kerr black hole solutions. The extremal limit with non-zero angular momentum has non-zero surface area and zero temperature. When $a = 0$, the above solution describes spherically symmetric black holes. If we set

$$
\begin{aligned}
r &\equiv \rho + m(\cosh\alpha\cosh\beta - 1), & r_+ &= m(1 + \cosh\alpha\cosh\beta) \\
r_- &= m(\cosh\alpha\cosh\beta - 1), & r_0 &= -m \sinh\alpha \sinh\beta \\
Q &= m(\sinh\alpha\cosh\beta - \sinh\beta\cosh\alpha), & P &= m(\sinh\alpha\cosh\beta + \sinh\beta\cosh\alpha)
\end{aligned}
\tag{62}
$$

the resulting expression coincides with the solution for (47) obtained from (23) by the duality transformation (39).

5. Discussion

We started with black hole solutions of 6D Kaluza–Klein theory and found interesting connections of those solutions with string theories. Two kinds of string theories are mainly discussed. One is 5D bosonic string theory compactified into four dimensions, and the other is 4D heterotic string theory with toroidal compactification. Actually, black hole solutions of 6D Kaluza–Klein theory and of 5D string theory can be read off from those of the heterotic string theory. Black hole solutions of 5D string theory can also be embedded into the closed superstring theory. It is not surprising that such connection exists between Kaluza–Klein black hole solutions and string theories. Many of supergravity theories can be obtained by dimensional reduction of the underlying supergravity theory of the closed superstring theory or the heterotic string theory with consistent truncation of some fields. Thus the embedding of massless fields of 5D string theory can be regarded as the embedding of the underlying supergravity theory into string theories.

Once black hole solutions of 5D string theory are embedded into the heterotic string theory or the closed string theory, we can generate other solutions using T-duality transformations. However, it is not clear that the transformed solutions are also exact solutions of the underlying string theory, since the leading-order duality transformation can receive α' corrections. It remains to be seen whether a similar argument of exactness can be given to the transformed solutions as Horowitz and Tseytlin did.

As this work was completed, we received a preprint by Cvetič *et al* [13] which worked out non-extremal solutions of 6D Kaluza–Klein theory independently.

Note added. The publication of the paper has been delayed by the author's personal situation. Since the paper has been distributed through the hep-th bulletin board and the author can claim the independence of the work, the author thinks that it is worthwhile to publish this paper.

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